Basics of group actions and Fuchsian groups; Fundamental domains; Dirichlet polygons

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MATH4900E Presentation 6, 9th November 2020

Outline

Outline

Introduction

Group actions:

A group homomorphism of a given group into the group of transformations of the space.

Definition (Left action)

A group G is said to act on a set X when there is a map $\zeta : G \times X \to X$ such that the following conditions hold for all elements $x \in X$:

1. $\zeta(id, x) = x$ where id is the identity element of G.
2. $\zeta(g, \zeta(h, x)) = \zeta(gh, x)$ for all $g, h \in G$.

Here, G is called a transformation group, ζ is called the group action.

Introduction

Type of group actions:

In topological space X, there are four actions of G:

1. Wandering
   If any $x \in X$ has a neighbourhood U such that $\{\ g \in G \mid g \cap U \neq \emptyset\}$ is finite.
2. **Properly discontinuously**

3. Proper
   If G is a topological group and the map from $G \times X \to X \times X : (g,\ x) \mapsto (g \cdot x,\ x)$ is proper.
4. Covering space action
   If any $x \in X$ has a neighbourhood U such that $\{\ g \in G \mid g \cdot U \cap U \neq \emptyset\ \} = \{id\}$

Introduction

Preliminaries on group actions:

1. Discreteness

2. Orbits

3. Stabilizer

Introduction

Recall:

Möb($\mathbb{H}$) and Möb($\mathbb{D}$) are groups (which are under composition). The collection of those Möbius transformations form a group.

General linear group: GL(2, $\mathbb{R}$)

Special linear group: SL(2, $\mathbb{R}$)

Projective special linear group: PSL(2, $\mathbb{R}$)

$\{$a, b, c, d $\in \mathbb{R}$ $\}$

Introduction

Recall:

$\text{Möb}(\mathbb{H}) := \{\gamma : z \mapsto \frac{az+b}{cz+d} \mid ad - bc = 1, a, b, c, d \in \mathbb{R}\}$ satisfies:

(a) Each $\gamma \in$ Möb($\mathbb{H}$) is an isometry.
$(d_{\mathbb{H}}(\gamma(z_1), \gamma(z_2)) = d_{\mathbb{H}}(z_1, z_2))$

(b) Möb($\mathbb{H}$) is a group, i.e.:

(i) Exists an identity element id. $(id(z) = z, \forall z \in \mathbb{H})$

(ii) $\gamma_1, \gamma_2 \in \text{Möb}(\mathbb{H}) \Rightarrow \gamma_1 \circ \gamma_2 \in \text{Möb}(\mathbb{H})$ (Not abelian)

(iii) $\forall \gamma \in \text{Möb}(\mathbb{H}) \Rightarrow \exists \gamma^{-1} \in \text{Möb}(\mathbb{H})$ $(\gamma \circ \gamma^{-1} = \gamma^{-1} \circ \gamma = id)$

(iv) $\gamma_1, \gamma_2, \gamma_3 \in \text{Möb}(\mathbb{H}) \Rightarrow (\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$

Discreteness

Discreteness is important in geometry, topology and metric spaces.

Metric space

A mathematical space on which it is possible to define the distance between two points in the space.

Let $d(x, y)$ be the distance between from x to y.

1: $d(x, y) > 0$ if $x \neq y$; $d(x, x) = 0$

2: $d(x, y) = d(y, x)$

3: $d(x, y) \leq d(x, z) + d(z, y)$

Discreteness

Examples of metric spaces:

i. $\mathbb{R}^n$ with the Euclidean metric

$$d((x_1, ..., x_n), (y_1, ..., y_n))$$

$$= ||(x_1, ..., x_n) - (y_1, ..., y_n)||$$

$$= \sqrt{|x_1 - y_1|^2 + ... + |x_n - y_n|^2}$$

ii. the upper half-plane $\mathbb{H}$ with the metric $d_{\mathbb{H}}$ that we defined in our last presentation, i.e.

$$d_{\mathbb{H}}(z, z') = inf\{ \text{ length}_{\mathbb{H}}(\sigma) \mid \sigma \text{ is a piecewise continuously differentiable path with end-points z and z'}\}$$

Discreteness

Metric space

Let (X, d) be a metric space. A subset $Y \subset X$ is discrete if every point $y \in Y$ is isolated.

Definition

A point $y \in Y$ is isolated if there exist $\delta > 0$ such that if $y' \in Y$ and $y' \neq y$, then $d(y, y') > \delta$.

Discreteness

Examples:

1. In any metric space, a single point $\{x\}$ is discrete.
2. The set of rationals $\mathbb{Q}$ is not a discrete subgroup of $\mathbb{R}$ since there are infinitely many distinct rationals arbitrarily close to any given rational.

Discreteness

Two Möbius transformations of $\mathbb{H}$ are close if the coefficients $(a,\ b,\ c,\ d)$ defining them are close.

But different coefficients $(a,\ b,\ c,\ d)$ can give the same Möbius transformations.

Recall:
Möbius transformation $\gamma(z) = \frac{az+b}{cz+d}$ is normalised if $ad - bc = 1$.
But,
if $\gamma(z) = \frac{az+b}{cz+d}$ is normalised, then $\gamma(z) = \frac{-az-b}{-cz-d}$ is also normalised.

Discreteness

The normalised Möbius transformations of $\mathbb{H}$ given by

$$\gamma_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$$

and

$$\gamma_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$$

If either $(a_1,\ b_1,\ c_1,\ d_1)$ and $(a_2,\ b_2,\ c_2,\ d_2)$ are close or $(a_1,\ b_1,\ c_1,\ d_1)$ and $(-a_2,\ -b_2,\ -c_2,\ -d_2)$ are close, then $\gamma_1(z)$ and $\gamma_2(z)$ are close.

Discreteness

Formula:

$$\mathrm{d}_{\text{Möb}}(\gamma_1, \gamma_2) = \min\{||(a_1, b_1, c_1, d_1) - (a_2, b_2, c_2, d_2)||, \\ ||(a_1, b_1, c_1, d_1) - (-a_2, -b_2, -c_2, -d_2)||\}$$

Think of Möbius transformations of $\mathbb{H}$ being close if they 'look close'.

Same as Möbius transformations of $\mathbb{D}$.

Discreteness

Definition

Let X be a subset of $\text{Möb}(\mathbb{H})$.

Then $\gamma \in X$ is isolated if there exist $\delta > 0$ such that $\forall \gamma' \in X - \{\gamma\}$, we have $d_{\text{Möb}}(y, y') > \delta$.

We say that a subset $X \subset \text{Möb}(\mathbb{H})$ is discrete if every $\gamma \in X$ is isolated.

Remark:
We could equally well work with isometries of $(\mathbb{D}, d_{\mathbb{D}})$ or any other model of hyperbolic space.

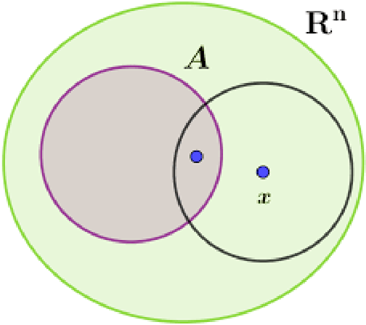
Discreteness

Definition

A subgroup $G \subset \mathrm{SL}(2, \mathbb{R})$ is a discrete group if G has no accumulation points in SL(2,$\mathbb{R}$).

Accumulation points

x is said to be an accumulation point in A if every open set containing x contains at least one other point from A.



Discreteness

Definition

A subset Z of $\mathbb{H}$ is discrete if for each $z \in Z$, there exists some $\varepsilon > 0$ so that $B(z,\ \varepsilon) \bigcap Z = \{z\}$, where

$$B(z,\ \varepsilon) = \{w \in \mathbb{H} \mid d_{\mathbb{H}}(z,\ w) < \varepsilon\}$$

is the open hyperbolic disc with hyperbolic centre z and hyperbolic radius ε.

That is same as each point of Z can be isolated from all the other points of Z.

Discreteness

Let Γ be a subgroup of Möb($\mathbb{H}$), and suppose Γ is not discrete. That is, there is some $z \in \mathbb{H}$ so that the set $\Gamma(z)$ is not a discrete subset of $\mathbb{H}$.

By the definition of discreteness, there exists an element $\gamma(z)$ of $\Gamma(z)$ so that for each $\varepsilon > 0$, the set $\Gamma \bigcap B(\gamma(z),\ \varepsilon)$ contains a point other than $\gamma(z)$.

For each $n \in \mathbb{N}$, choose an element γ_n of Γ so that $\gamma_n(z) \neq \gamma(z)$ and so that

$$\gamma_n(z) \in \Gamma(z) \bigcap B(\gamma(z),\ \frac{1}{n}).$$

As $n \to \infty$, we have that $\mathrm{d}_{\mathbb{H}}(\gamma(z),\ \gamma_n(z)) \to 0$. Pass to a subsequence of $\{\gamma_n\}$, called $\{\gamma_n\}$ to avoid the proliferation of subscripts, so that the $\gamma_n(z)$ are distinct. We now have a sequence $\{\gamma_n\}$ of distinct elements of Γ so that $\{\gamma_n\}$ converges to $\gamma(z)$.

Discreteness

Lemma 1

Let Γ be a subgroup of $\text{Möb}(\mathbb{H})$. Γ contains a sequence of distinct elements converging to an element μ of $\text{Möb}(\mathbb{H})$ if and only if Γ contains a sequence of distinct elements converging to the identity.

Proposition 1

Let Γ be a discrete subgroup of $\text{Möb}(\mathbb{H})$. If X is a subgroup of Γ, then X is discrete.

Conversely, there are a few special cases in which the discreteness of a subgroup of Γ implies the discreteness of Γ. We begin considering subgroups of $\text{Möb}^+(\mathbb{H})$ with discrete normal subgroups.

Discreteness

Proposition 2

Let Γ be a discrete subgroup of $\text{Möb}^+(\mathbb{H})$ and let X be a non-trivial normal subgroup of Γ. If X is discrete, then Γ is discrete.

Proof:
To prove this proposition, we will use the contrapositive.

Suppose that Γ is not discrete and let $\{\gamma_n\}$ be a sequence of distinct elements of Γ coverging to the identity.

Choose some element μ of X, other than the identity, and consider the sequence $\{\gamma_n^{-1} \circ \mu \circ \gamma_n\}$.

Discreteness

Observe that $\{\gamma_n^{-1} \circ \mu \circ \gamma_n\}$ is a sequence of elements of X.

Since $\{\gamma_n\}$ converges to the identity, we have that $\{\gamma_n^{-1}\}$ converges to the identity as well, and so $\{\gamma_n^{-1} \circ \mu \circ \gamma_n\}$ converges to μ.

Then since γ_n are distinct and are converging to the identity, $\gamma_n^{-1} \circ \mu \circ \gamma_n$ are distinct.

Therefore, X is not discrete.

Discreteness

Proposition 3

Let Γ be a subgroup of $\text{Möb}(\mathbb{H})$, and let X be a finite index subgroup of Γ. If X is discrete, then Γ is discrete.

Proof: First, we need to express Γ as a coset decomposition with respect to X, that is:

$$\Gamma = \bigcup_{k=0}^{p} \alpha_k X,$$

where $\alpha_0, \cdots, \alpha_p$ are elements of Γ.

Discreteness

Suppose that Γ is not discrete, and let $\{\gamma_n\}$ be a sequence of distinct elements of Γ converging to the identity.
For n, we can write $\gamma_n = \alpha_{k_n}\mu_n$, where $0 \leq k_n \leq p$ and $\mu_n \in X$.
Since there are infinitely many elements in the sequence, there is some fixed q satisfying $0 \leq q \leq p$, so that $k_n = q$ for infinitely many n.
So, consider the subsequence $\{\gamma = \alpha_q\mu_m\}$ consisting of those elements of the sequence for which $k_n = q$.
Since $\{\gamma_m\}$ converges to the identity, we have that $\{\alpha_q\mu_m\}$ converges to the identity as well. Hence, we have that $\{\mu_m\}$ converges to α_q^{-1}.
By the lemma, X is not discrete, a contradiction.

Orbits

Let Γ be a discrete subgroup of $\text{Möb}(\mathbb{H})$.

Definition

Let $z \in \mathbb{H}$. The orbit $\Gamma(z)$ of z under Γ is the set of all points of $\mathbb{H}$ that we can reach by applying elements of Γ to z:

$$\Gamma(z) := \{\gamma(z) \mid \gamma \in \Gamma\}.$$

Orbits

Example:

Let $\Gamma(z) = \{\gamma(z) = \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1\}$.
Let z=i, then we have,

$$\Gamma(i) = \{\frac{ai+b}{ci+d} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1\}$$

Example:

Let $\Gamma(z) = \{\gamma(z) = \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1\}$. Let $z = 0 \in \partial\mathbb{H}$, then we have,

$$\Gamma(0) = \{\frac{b}{d} \mid ad - bc = 1\}$$
$$= \mathbb{Q} \cup \{\infty\}.$$

An irrational point on $\mathbb{R}$ can always be arbitrarily well approximated by rationals.

Stabilizer

Definition

Let $z \in \mathbb{H}$. The stabilizer Γ_z of z under Γ is defined as:

$$\Gamma_z := \{\gamma \in \Gamma \mid \gamma(z) = z\}.$$

Theorem

Let Γ be a subgroup of $\text{Möb}(\mathbb{H})$. If Γ is discrete, then the stabilizer

$$\Gamma_z := \{\gamma \in \Gamma \mid \gamma(z) = z\}.$$

is finite for every $z \in \mathbb{H}$.

But, the converse of this theorem does not hold.

Stabilizer

Proof by using an example:

Consider the subgroup

$$\Gamma = \{m_\lambda(z) = \lambda z \mid \lambda > 0\}$$

of $\text{Möb}(\mathbb{H})$.

Then Γ is not a discrete subgroup of $\text{Möb}(\mathbb{H})$.

However, the stabilizer Γ_z is trivial for every $z \in \mathbb{H}$.

Outline

Fuchsian groups

Definition

A Fuchsian group is a discrete subgroup of either Möb($\mathbb{H}$) or Möb($\mathbb{D}$).

Examples.

1. Any finite subgroup of Möb($\mathbb{H}$) or Möb($\mathbb{D}$) is a Fuchsian group because any finite subset of any metric space is discrete.
2. As a specific example in the upper half-plane, let

$$\gamma_\theta(z) = \frac{cos(\theta/2)z + sin(\theta/2)}{-sin(\theta/2)z + cos(\theta/2)}$$

   be a rotation around i.
   Let $q \in \mathbb{N}$. Then $\{\gamma_{2\pi j/q} \mid 0 \leq j \leq q-1\}$ is a finite subgroup.

Fuchsian groups

3. The subgroup of integer translations $\{\gamma_n(z) = z + n \mid n \in \mathbb{Z}\}$ is a Fuchsian group. The subgroup of all translations $\{\gamma_b(z) = z + b \mid b \in \mathbb{R}\}$ is not a Fuchsian group as it is not discrete.
4. The subgroup $\Gamma = \{\gamma_n(z) = 2^n z \mid n \in \mathbb{Z}\}$ is a Fuchsian group.
5. The subgroup $\Gamma = \{id\}$ containing only the identity Möbius transformation is a Fuchsian group. We call it the trivial Fuchsian group.
6. If Γ is a Fuchsian group and $\Gamma_1 < \Gamma$ is a subgroup, then Γ_1 is a Fuchsian group.

Fuchsian groups

7. One of the most important Fuchsian groups is the modular group $\mathrm{PSL}(2, \mathbb{Z})$. This is the group given by Möbius transformations of $\mathbb{H}$ of the form

$$\gamma(z) = \frac{az+b}{cz+d}, \ a, \ b, \ c, \ d \in \mathbb{Z}, \ ad - bc = 1$$

8. Let $q \in \mathbb{N}$. Define

$$\Gamma_q = \{\frac{az+b}{cz+d} \mid a, \ b, \ c, \ d \in \mathbb{Z}, \ ad{-}bc = 1, \ b, \ c \text{ are divisible by } q\}.$$

This is called the level q modular group or the congruence subgroup of order q.

Fuchsian groups

Example (3):
Let $\gamma(z) = z + n$. Then $\Gamma = \{..., \gamma^{-1}, id, \gamma, \gamma^2, ...\}$

Note that $\gamma^a(z) = z + an$ and $\gamma^b(z) = z + bn$ for $a, b \in \mathbb{Z}$ for $a \neq b$,

$$\gamma^a(z) = \frac{(1)z + an}{(0)z + 1}$$

and

$$\gamma^b(z) = \frac{(1)z + bn}{(0)z + 1}$$

Thus $\mathrm{d}_{\text{Möb}}(\gamma^a, \gamma^b) = \min\{|an - bn|, |an + bn|\}$

$$\geq \frac{|n|}{2} > 0, \text{ for } a \neq b$$

Fuchsian groups

Example (7):

Let $\Gamma = \{\ \gamma(z) = \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{Z},\ ad-bc=1\}$

First, need to prove it is a group.

Let

$$\gamma_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}\ ,\ a_1, b_1, c_1, d_1 \in \mathbb{Z},\ a_1 d_1 - b_1 c_1 = 1$$

and

$$\gamma_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}\ ,\ a_2, b_2, c_2, d_2 \in \mathbb{Z},\ a_2 d_2 - b_2 c_2 = 1$$

Thus $\mathrm{d}_{\text{Möb}}(\gamma_1, \gamma_2) \geq 1 > 0$

Therefore Γ is a Fuchsian group.

Fuchsian groups

Proposition 4

Let Γ be a subgroup of $\text{Möb}(\mathbb{H})$. The following are equivalent:

i. Γ is a discrete subgroup of $\text{Möb}(\mathbb{H})$ (i.e. Γ is a Fuchsian group);

ii. the identity element of Γ is isolated.

Proof:

i $\Rightarrow$ ii: is trivial from the definition.

ii $\Rightarrow$ i: Given $\gamma \in \Gamma$, we can consider the continuous map:

$$\begin{cases} \mathrm{B}_{\text{Möb}}(\gamma,\ \varepsilon) & \xrightarrow{f} f(\mathrm{B}_{\text{Möb}}(\gamma,\ \varepsilon)) \subseteq \text{Möb}(\mathbb{H}) \\ \gamma' & \xmapsto{f} \gamma^{-1}\gamma' \text{ (Multiply by } \gamma^{-1}) \end{cases}$$

where $\mathrm{B}_{\text{Möb}}(\gamma,\ \varepsilon) = \{\gamma' \mid \mathrm{d}_{\text{Möb}}(\gamma, \gamma') < \varepsilon\}$ for some $\varepsilon > 0$.

Fuchsian groups

Since id is isolated, we can choose $\delta > 0$ with $B_{\text{Möb}}(id,\ \delta) \cap \Gamma = \{id\}$.

Since f is a homeomorphism onto its image and $f(\gamma) = id$, then we can choose $\varepsilon > 0$ which is small enough that $f(B(\gamma,\ \varepsilon)) \subset B(id,\ \delta)$

Then, since $f(B(\gamma,\ \varepsilon)) \cap \Gamma = \{id\}$, therefore $B(\gamma,\ \varepsilon) \cap \Gamma = \{\gamma\}$

Fuchsian groups

Theorem (Jørgensen's inequality)

Let $\Gamma \subseteq \text{Möb}(\mathbb{H})$ be generated by two elements $\gamma_1, \gamma_2 \in \text{Möb}(\mathbb{H})$. A necessary condition for Γ to be Fuchsian is that

$$\max\{||\gamma_1 - id||,\ ||\gamma_2 - id||\} > \frac{7}{50}$$

Refer to *The Geometry of Discrete Groups*, P. 107.

Theorem (Shimizu's Lemma)

If Γ is a Fuchsian group and $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, then for any $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have either $c = 0$ or $|c| \geq 1$.

Fuchsian groups

Proposition 5

Let Γ be a subgroup of $\text{Möb}(\mathbb{H})$. Then the following are equivalent:

i. Γ is a Fuchsian group;

ii. For each $z \in \mathbb{H}$, the orbit $\Gamma(z)$ is a discrete subset of $\mathbb{H}$.

Suppose this statement holds in the case of $\mathbb{D}$.

Fuchsian groups

Example:

Let $\Gamma = \{\gamma_n \mid \gamma_n(z) = 2^n z,\ n \in \mathbb{Z}\}$. Fix $z \in \mathbb{H}$. Then the orbit of z is

$$\Gamma(z) = \{2^n z \mid n \in \mathbb{Z}\}.$$

Observe that the points $2^n z$ lie on the (Euclidean) straight line through the origin inclined at angle $arg(z)$. Fix $2^n z$ and let $\delta = 2^{n-1}|z|$.

Then, $|2^m z - 2^n z| \geq \delta$ whenever $m \neq n$.

Hence $\Gamma(z)$ is discrete.

Fuchsian groups

Example:

Fix $k > 0,\ k \neq 1$. Consider the subgroup of Möb($\mathbb{H}$) generated by the Möbius transformations of $\mathbb{H}$ given by

$$\gamma_1(z) = z + 1,\ \gamma_2(z) = kz.$$

First consider the orbit $\Gamma(i)$ of i.
Assume that $k > 1$, then observe that

$$\gamma_2^{-n}\gamma_1^m\gamma_2^n(i) = i + \frac{m}{k^n}.$$

Assume that $0 < k < 1$, then observe that

$$\gamma_2^n\gamma_1^m\gamma_2^{-n}(i) = i + mk^n.$$

Choose an arbitrarily large n. Then i is not an isolated point of the orbit $\Gamma(i)$. Hence $\Gamma(i)$ is not discrete.

Properly discontinuous

Definition

The group Γ acts properly discontinuously on $\mathbb{H}$ if $\forall z_0 \in \mathbb{H}$ and any compact set $K \subseteq \mathbb{H}$, the set $\{ \gamma \in \Gamma \mid \gamma(z_0) \in K\}$ is finite.

Note that we could replace K by closed ball which is:

$$\overline{\mathrm{B}(p,\ r)} = \{ z \in \mathbb{H} \mid \mathrm{d}(p,\ z) \leq r\}$$

for any $\varepsilon > 0$.

Properly discontinuous

Lemma 2

Let $\Gamma \subseteq \text{Möb}(\mathbb{H})$ be a subgroup acting properly discontinuously on $\mathbb{H}$.

Let $z_0 \in \mathbb{H}$ be fixed by $\gamma_0 \in \Gamma$. $(\gamma_0(z_0) = z_0)$

Then $\exists$ neighbourhood $W \ni z_0$ such that no other point in W is fixed by a non-identity element of Γ.

Properly discontinuous

Proof by contradiction:

Assume for a contradiction $\begin{cases} \mathrm{z}_n & \to z_0 \ (n \geq 1) \\ \exists \gamma_n & \in \Gamma - \{e\},\ \gamma_n(z_n) = z_n \end{cases}$

Therefore, $\forall \varepsilon > 0$, $\exists N_1$ such that $\forall n > N_1$, $\mathrm{d}(z_n,\ z_0) < \varepsilon$.

Since :

i $\overline{\mathrm{B}(z_0,\ 2\varepsilon)} = \{\ z \in \mathbb{H} \mid \mathrm{d}(z,\ z_0) \leq 2\varepsilon\}$ is a compact.

ii Γ acts discontinuously on $\mathbb{H}$.

$\Rightarrow \{\ \gamma \in \Gamma \mid \gamma(z_0) \in \overline{\mathrm{B}_{2\varepsilon}(z_0)}\}$ is finite.

Therefore, $\exists N_2 \geq 1$, $\forall n > N_2$, $\mathrm{d}(\gamma_n(z_0),\ z_0) > 2\varepsilon$.

For $n > \max\{N_1,\ N_2\}$: $\begin{cases} \mathrm{d}(\mathrm{z}_n,\ z_0) < \varepsilon \\ \mathrm{d}(\gamma_n(z_0),\ z_0) > 2\varepsilon \end{cases}$

Properly discontinuous

Hence:

$$\begin{aligned} 2\varepsilon < \mathrm{d}(\gamma_n(z_0),\ z_0) &\leq \mathrm{d}(\gamma_n(z_0),\ \gamma_n(z_n)) + \mathrm{d}(\gamma_n(z_n),\ z_0) \\ &= \mathrm{d}(z_0,\ z_n) + \mathrm{d}(z_n,\ z_0) \\ &= 2\mathrm{d}(z_0,\ z_n) \\ &< 2\varepsilon \end{aligned}$$

Contradiction arises.

Properly discontinuous

Corollary

If Γ acts properly discontinuously on $\mathbb{H}$, then we can find $z_0 \in \mathbb{H}$ which is not fixed by any $\gamma \in \Gamma - \{id\}$.

Lemma 3

For $z_0 \in \mathbb{H}$ and a compact set $K \subseteq \mathbb{H}$:

$$E = \{\, \gamma(z) = \frac{az+b}{cz+d} \mid ad{-}bc = 1,\ a,\ b,\ c, d \in \mathbb{R}, \gamma(z_0) \in K\} (\subseteq \mathbb{R}^4)$$

is compact.

Properly discontinuous

Proof:

Since K is compact, we can choose $k_1,\ k_2 > 0$:

$$\begin{cases} k_1 \leq \mathrm{Im}(\gamma(z_0)) = \frac{\mathrm{Im}(z_0)}{|cz_0+d|^2} \qquad (1) \\ \\ k_2 \geq \gamma(z_0) = |\frac{az_0+b}{cz_0+d}| \qquad (2) \end{cases}$$

Thus, $$\begin{cases} |cz_0 + d| \leq \sqrt{\frac{\mathrm{Im}(z_0)}{k_1}} = c_1 \qquad (3) \\ \\ |az_0 + b| \leq k_2\sqrt{\frac{\mathrm{Im}(z_0}{k_1}} = c_2 \qquad (4) \end{cases}$$

From these constraints on $a,\ b,\ c,\ d$, we can deduce that E is bounded. Clearly it is also closed, and thus compact.

Properly discontinuous

Proposition 6

Let Γ be a subgroup of $\text{Möb}(\mathbb{H})$. The following are equivalent:

(i) Γ is Fuchsian;

(ii) Γ acts properly discontinuously on $\mathbb{H}$.

Proof:

(i) $\Rightarrow$ (ii):

Let $z_0 \in \mathbb{H}$ and $K \subseteq \mathbb{H}$ be compact.

Since $\{ \gamma \in \Gamma \mid \gamma(z_0) \in K\} = \{ \gamma \in \text{Möb}(\mathbb{H}) \mid \gamma(z_0) \in K \} \cap \Gamma$, the intersection is finite. (Γ acts properly discontinuously.)

Properly discontinuous

(ii) $\Rightarrow$ (i):
Assume Γ acts properly discontinuously on $\mathbb{H}$.

By the **Corollary**:
$\exists z \in \mathbb{H}$ such that if $\gamma \in \Gamma$ and $\gamma(z) = z \Rightarrow \gamma = id$.

Assume for a contradiction, Γ is not discrete.
Therefore, we can find a sequence

$$\begin{cases} \gamma_n \in \Gamma, n \geq 1 \\ \gamma_n \to id \text{ (Without loss of generality)} \end{cases}$$

In particular, $\begin{cases} \gamma_n(z) \to z \text{ as } n \to \infty \\ \gamma_n(z) \neq z, n \geq 1 \end{cases}$

Thus, $\forall \varepsilon > 0, \{ \gamma \in \Gamma \mid \gamma(z) \in \overline{B(z,\ \varepsilon)}\}$ is infinite.

Properly discontinuous

Proposition 7

Let Γ be a subgroup of Möb($\mathbb{H}$). Then Γ acts properly discontinuously on $\mathbb{H}$ if and only if for all $z \in \mathbb{H}$, $\Gamma(z)$, the orbit of z, is a discrete subset of $\mathbb{H}$.

Proof:
($\Rightarrow$): Suppose Γ acts properly discontinuously on $\mathbb{H}$.
Then Γ is a Fuchsian group, and hence $\Gamma(z)$ is a discrete subset of $\mathbb{H}$.

Properly discontinuous

($\Leftarrow$): Prove by contradiction:
Suppose Γ does not act properly discontinuously on $\mathbb{H}$.
Hence by the theorem, Γ is not discrete.
Then using the previous sequence, we can see that the orbit of z is not discrete.

Summary

Group action:

1. Discreteness: $d_{\text{Möb}(\mathbb{H})}(\gamma_1, \ \gamma_2) > \delta > 0$.
2. Orbits: $\Gamma(z) = \{\gamma \in \Gamma \mid \gamma(z)\}$.
3. Stabilizer: $\Gamma_z = \{\gamma \in \Gamma \mid \gamma(z) = z\}$.
4. Properly discontinuous: $\forall z_0 \in \mathbb{H}$ and any compact set $K \subseteq \mathbb{H}$, the set $\{ \ \gamma \in \Gamma \mid \gamma(z_0) \in K\}$ is finite.

Fuchisan group:

1. is a discrete subgroup of Möb($\mathbb{H}$) or Möb($\mathbb{D}$).
2. identity element is isolated.
3. orbit is discrete subset of $\mathbb{H}$.
4. acts properly discontinuously on $\mathbb{H}$.

Outline

Open and closed subsets

Definition

A subset Y $\subset \mathbb{H}$ is said to be **open** if $\forall y \in Y, \exists\, \varepsilon > 0$ such that the open ball $B_\varepsilon(y) = \{z \in \mathbb{H} \,|\, d_{\mathbb{H}}(z, y) < \varepsilon\}$ of radius ε and centre y is contained in Y.
A subset Y $\subset \mathbb{H}$ is said to be **closed** if its complement $\mathbb{H} \setminus Y$ is open.

Examples:

1. The subset $\{z \in \mathbb{H} \,|\, 0 < Re(z) < 1\}$ is open.
2. The subset $\{z \in \mathbb{H} \,|\, 0 \leq Re(z) \leq 1\}$ is closed.
3. The subset $\{z \in \mathbb{H} \,|\, 0 < Re(z) \leq 1\}$ is neither open nor closed.
4. The subset $\emptyset$ is both open and closed.

Open and closed subsets: Remark

Note that hyperbolic circles are Euclidean circles (albeit with different radii and centres).

Fact:
Let $C = \{w \in \mathbb{H} \mid d_{\mathbb{H}}(z, w) = r\}$ be a hyperbolic circle with centre $z \in \mathbb{H}$ and radius $r > 0$. Let $z = x_0 + iy_0$. Then C is a Euclidean circle with centre $(x_0,\, y_0 \cosh r)$ and radius $y_0\sqrt{\cosh^2 r - 1} = y_0 \sinh r$.

Thus to prove a subset $Y \subset \mathbb{H}$ is open it is sufficient to find a Euclidean open ball around each point that is contained in Y .

In particular, the open subsets of $\mathbb{H}$ are the same as the open subsets of the (Euclidean) upper half-plane.

Closure

Definition

Let $Y \subset \mathbb{H}$ be a subset. Then the **closure** of Y is the smallest closed subset containing Y. We denote the closure of Y by $cl(Y)$.

Example

The closure of $\{z \in \mathbb{H} \,|\, 0 < Re(z) < 1\}$ and $\{z \in \mathbb{H} \,|\, 0 < Re(z) \leq 1\}$ is $\{z \in \mathbb{H} \,|\, 0 \leq Re(z) \leq 1\}$.

Properties of closed sets:

1. Any intersection of closed sets is closed.
2. The union of finitely many closed sets is closed.

Fundamental domain

Definition

Let Γ be a Fuchsian group. A **fundamental domain** F for Γ is an open subset of $\mathbb{H}$ such that:

(i) $\bigcup_{\gamma \in \Gamma} \gamma(cl(F)) = \mathbb{H}$

(ii) the images $\gamma(F)$ are pairwise disjoint; that is, $\gamma_1(F) \cap \gamma_2(F) = \emptyset$ if $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$.

Remark

Since both γ and γ^{-1} are continuous maps, $\gamma(cl(F)) = cl(\gamma(F))$. Thus F is a fundamental domain if every point lies in the closure of some image $\gamma(F)$ and if two distinct images do not overlap. We say that the images of F under Γ **tessellate** $\mathbb{H}$.

Example of Fuchsian group (I): Integer translations

The subgroup Γ of Möb($\mathbb{H}$) given by integer translations: $\Gamma_n(z) = \{\gamma_n \,|\, \gamma_n(z) = z + n,\, n \in \mathbb{Z}\}$ is a Fuchsian group.

Proof

Consider the set $F = \{z \in \mathbb{H} \,|\, 0 < Re(z) < 1\}$. This is an open set. Clearly if $Re(z) = a$, then $Re(\gamma_n(z)) = n + a$. Hence

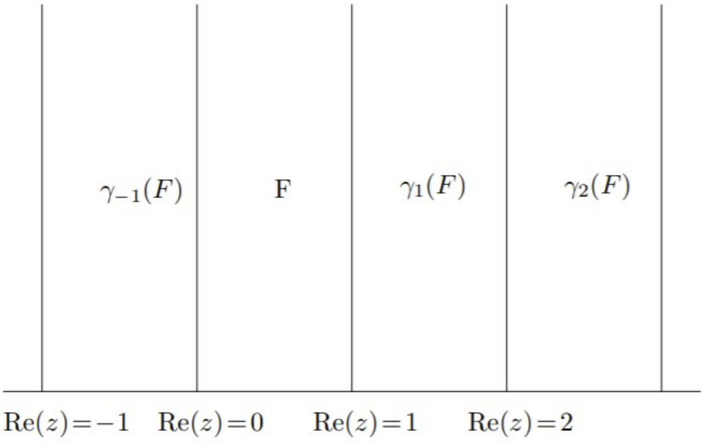
$$\gamma_n(F) = \{z \in \mathbb{H} \,|\, n < Re(z) < n + 1\}$$

and

$$\gamma_n(cl(F)) = \{z \in \mathbb{H} \,|\, n \le Re(z) \le n + 1\}$$

Hence $\mathbb{H} = \bigcup_{n\in\mathbb{Z}} \gamma_n(cl(F))$. It is also clear that if $\gamma_n(F)$ and $\gamma_m(F)$ intersect, then $n = m$. Hence F is a fundamental domain for Γ.

A fundamental domain and tessellation for $\Gamma = \{\gamma_n \mid \gamma_n(z) = z + n\}$



Example of Fuchsian group (II)

The subgroup $\Gamma = \{\gamma_n \,|\, \gamma_n(z) = 2^n z, n \in \mathbb{Z}\}$ of Möb($\mathbb{H}$) is a Fuchsian group.

Proof

Let $F = \{z \in \mathbb{H} \,|\, 1 < |z| < 2\}$. This is an open set. Clearly, if $1 < |z| < 2$ then $2^n < |\gamma_n(z)| < 2^{n+1}$. Hence

$$\gamma_n(F) = \{z \in \mathbb{H} \,|\, 2^n < |z| < 2^{n+1}\}$$

and

$$\gamma_n(cl(F)) = \{z \in \mathbb{H} \,|\, 2^n \leq |z| \leq 2^{n+1}\}$$

Hence $\mathbb{H} = \bigcup_{n\in\mathbb{Z}} \gamma_n(cl(F))$. It is also clear that if $\gamma_n(F)$ and $\gamma_m(F)$ intersect, then $n = m$. Hence F is a fundamental domain for Γ.

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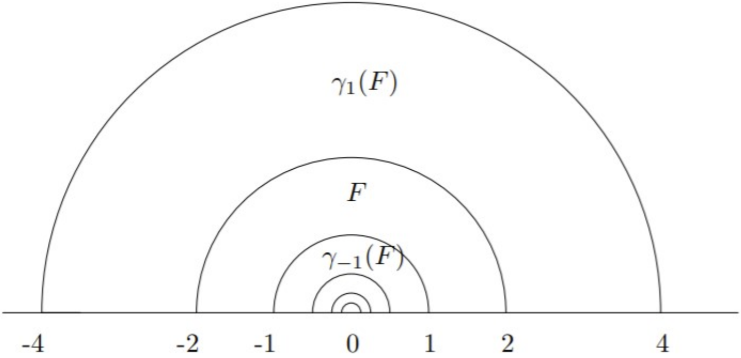
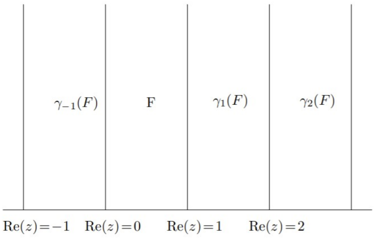
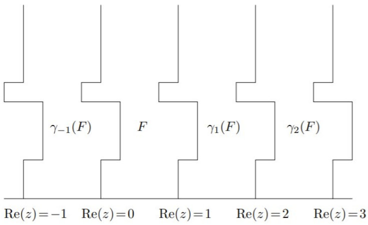


Uniqueness of Fundamental domains

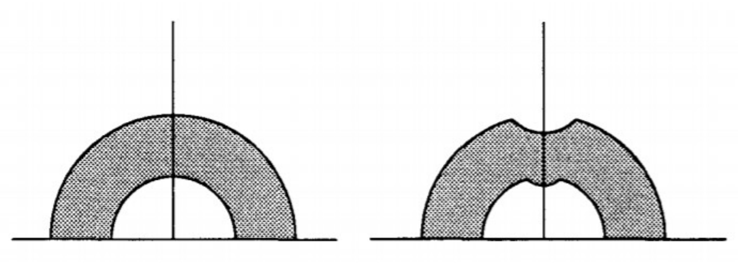
Suppose $\Gamma = \{\mathrm{Id}\}$, the trivial group containing just one element. In this case, $\mathbb{H}$ is the only fundamental domain for Γ. Now suppose $\Gamma \neq \{\mathrm{Id}\}$. A fundamental domain is not uniquely determined by a non-trivial Fuchsian group: an arbitary small perturbation gives another fundamental domain.





Fundamental domains are not unique - continued

Let Γ be the cyclic group generated by the transformation $z \to 2z$. The fundamental domains for Γ are:



Any two fundamental regions have the same area

Recall that: The boundary ∂F of a set F is defined to be the set $cl(F) \setminus int(F)$.
Here $cl(F)$ is the closure of F and $int(F)$ is the interior of F.

Proposition

Let F_1 and F_2 be two fundamental domains for a Fuchsian group γ, with $\mathrm{Area}_{\mathbb{H}}(F_1) < \infty$. Assume that $\mathrm{Area}_{\mathbb{H}}(\partial F_1) = 0$ and $\mathrm{Area}_{\mathbb{H}}(\partial F_2) = 0$. Then $\mathrm{Area}_{\mathbb{H}}(F_1) = \mathrm{Area}_{\mathbb{H}}(F_2)$.

Proof of Proposition

Since $\text{Area}_{\mathbb{H}}(\partial F_i) = 0$, $\text{Area}_{\mathbb{H}}(cl(F_i)) = \text{Area}_{\mathbb{H}}(F_i)\ \forall i = 1, 2$
Hence, we have:

$$cl(F_1) \supset cl(F_1) \cap (\bigcup_{\gamma \in \Gamma} \gamma(F_2)) = \bigcup_{\gamma \in \Gamma} (cl(F_1) \cap \gamma(F_2))$$

As F_2 is a fundamental domain, the sets $cl(F_1) \cap \gamma(F_2)$ are pairwise disjoint.
Hence, using the facts that

(i) the area of the union of disjoint sets is the sum of the areas of the sets,

(ii) Möbius transformations of $\mathbb{H}$ preserve area.

Proof of Proposition - continued

We have:

$$\begin{aligned} Area_{\mathbb{H}}(cl(F_1)) &\geq \sum_{\gamma\in\Gamma} Area_{\mathbb{H}}(cl(F_1)\cap\gamma(F_2)) \\ &= \sum_{\gamma\in\Gamma} Area_{\mathbb{H}}(\gamma^{-1}(cl(F_1))\cap F_2) \\ &= \sum_{\gamma\in\Gamma} Area_{\mathbb{H}}(\gamma(cl(F_1))\cap F_2) \end{aligned}$$

Since F_1 is a fundamental domain we have:

$$\bigcup_{\gamma\in\Gamma}\gamma(cl(F_1)) = \mathbb{H}$$

Proof of Proposition - continued

Hence

$$\sum_{\gamma\in\Gamma} \mathrm{Area}_{\mathbb{H}}(\gamma(cl(F_1)) \cap F_2) \geq \mathrm{Area}_{\mathbb{H}}\Big(\bigcup_{\gamma\in\Gamma} \gamma(cl(F_1)) \cap F_2\Big)$$
$$= \mathrm{Area}_{\mathbb{H}}(F_2)$$

Hence

$$\mathrm{Area}_{\mathbb{H}}(F_1) = \mathrm{Area}_{\mathbb{H}}(cl(F_1)) \geq \mathrm{Area}_{\mathbb{H}}(F_2)$$

Interchanging F_1 and F_2 in the above gives the reverse inequality.

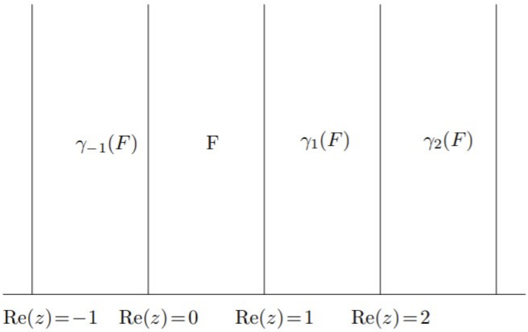
$$\mathrm{Area}_{\mathbb{H}}(F_2) = \mathrm{Area}_{\mathbb{H}}(cl(F_2)) \geq \mathrm{Area}_{\mathbb{H}}(F_1)$$

Hence $\mathrm{Area}_{\mathbb{H}}(F_1) = \mathrm{Area}_{\mathbb{H}}(F_2)$.

Points to note

The area of a fundamental region, if it is finite, is a numerical invariant of the group.

Integer translations are examples of a Fuchsian group with a fundamental domain of infinite area.



A Fuchsian group and its subgroup

Let Γ be a Fuchsian group and let $\Gamma_1 < \Gamma$ be a subgroup of Γ. Then Γ_1 is a discrete subgroup of the Möbius group Möb($\mathbb{H}$) and so is itself a Fuchsian group.

Definition

Let G be a group. A subset H of G is a **subgroup** of G if it satisfies the following properties:

- **Closure:** If $a, b \in H$, then $ab \in H$.
- **Identity:** The identity element of G lies in H.
- **Inverses:** If $a \in H$, then $a^{-1} \in H$.

Definition

The **index** of a subgroup H in a group G is the number of left cosets of H in G, or equivalently, the number of right cosets of H in G.

A Fuchsian group and its subgroup

Proposition

Let Γ be a Fuchsian group and suppose that Γ_1 is a subgroup of Γ of index n. Let

$$\Gamma = \Gamma_1\gamma_1 \cup \Gamma_1\gamma_2 \cup \cdots \cup \Gamma_1\gamma_n$$

be a decomposition of Γ into cosets of Γ_1. Let F be a fundamental domain for Γ. Then:

(i) $F_1 = \gamma_1(F) \cup \gamma_2(F) \cup \cdots \cup \gamma_n(F)$ is a fundamental domain for Γ_1;

(ii) if $\mathrm{Area}_{\mathbb{H}}(F)$ is finite then $\mathrm{Area}_{\mathbb{H}}(F_1) = n\mathrm{Area}_{\mathbb{H}}(F)$.

Summary: Fundamental domains

1. A Fuchsian group is a discrete subgroup of the group Möb($\mathbb{H}$) of all Möbius transformations of $\mathbb{H}$.
2. A subset $F \subset \mathbb{H}$ is a fundamental domain if, essentially, the images $\gamma(F)$ of F under the Möbius transformations $\gamma \in \Gamma$ tessellate (or tile) the upper half-plane $\mathbb{H}$.
3. The set $\{z \in \mathbb{H} \,|\, 0 < Re(z) < 1\}$ is a fundamental domain for the group of integer translations $\{\gamma_n(z) = z + n \,|\, n \in \mathbb{Z}\}$

Outline

Introduction to Dirichlet polygon

Each Fuchsian group possesses a fundamental domain.
The purpose of the following slides is to give a method for constructing a fundamental domain for a given Fuchsian group.
The fundamental domain that we construct is called a **Dirichlet polygon**.

There are other methods for constructing fundamental domains that, in general, give different fundamental domains than a Dirichlet polygon; such an example is the Ford fundamental domain.

The construction given below is written in terms of the upper half-plane $\mathbb{H}$. The same construction works in the Poincaré disc $\mathbb{D}$.

Dirichlet polygon

Definition

Let C be a geodesic in $\mathbb{H}$. Then C divides $\mathbb{H}$ into two components. These components are called **half-planes**.

Example 1:
The imaginary axis determines two half-planes:
$\{z \in \mathbb{H} \mid Re(z) < 0\}$ and $\{z \in \mathbb{H} \mid Re(z) > 0\}$.

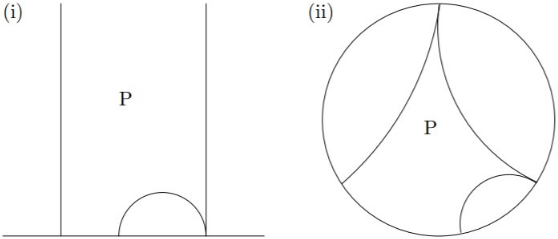
Example 2:
The geodesic given by the semi-circle of unit radius centred at the origin also determines two half-planes (although they no longer look like Euclidean half-planes): $\{z \in \mathbb{H} \mid |z| < 1\}$ and $\{z \in \mathbb{H} \mid |z| > 1\}$.

Convex hyperbolic polygon

Definition

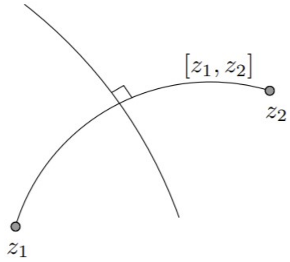
A convex hyperbolic polygon is the intersection of a finite number of halfplanes.

It is possible that an edge of a hyperbolic polygon to be an arc of the circle at infinity. For example, a polygon with one edge on the boundary (i) in the upper half-plane, and (ii) in the Poincaré disc.



Perpendicular bisectors

Let $z_1, z_2 \in \mathbb{H}$. Recall that $[z_1, z_2]$ is the segment of the unique geodesic from z_1 to z_2. The perpendicular bisector of $[z_1, z_2]$ is defined to be the unique geodesic perpendicular to $[z_1, z_2]$ that passes through the midpoint of $[z_1, z_2]$.



Perpendicular bisectors: Proposition

Proposition

Let $z_1, z_2 \in \mathbb{H}$. The set of points $\{z \in \mathbb{H} \mid d_{\mathbb{H}}(z, z_1) = d_{\mathbb{H}}(z, z_2)\}$ that are equidistant from z_1 and z_2 is the perpendicular bisector of the line segment $[z_1, z_2]$.

Proof

Without loss of generality (by applying a Möbius isometry, if necessary), we can write:

$$\begin{cases} z_1 = i \\ z_2 = ir^2 \, (r > 1) \end{cases}$$

There is no loss in generality to assume that $r > 1$, since we can apply the Möbius transformation $z \mapsto -\frac{1}{z}$, if required.

Proof of proposition - continued

Recall that:
Let $a \leq b$. Then the hyperbolic distance between ia and ib is $log\frac{b}{a}$. Moreover, the vertical line joining ia to ib is the unique path between ia and ib with length $log\frac{b}{a}$; any other path from ia to ib has length strictly greater than $log\frac{b}{a}$.

Using the above fact, it follows that the mid-point of $[i, ir^2]$ is at the point ir. It is clear that the unique geodesic through ir that meets the imaginary axis at right-angles is given by the semi-circle of radius r centred at 0.

Proof of proposition - continued

Recall that:

$$\cosh d_{\mathbb{H}}(z, w) = 1 + \frac{|z - w|^2}{2 \operatorname{Im} z \operatorname{Im} w}$$

In our setting this implies that:

$$|z - i|^2 = \frac{|z - ir^2|^2}{r^2}$$

This simplifies to $|z| = r$, i.e. z lies on the semicircle of radius r, centred at 0.

Example

Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2, z_1, z_2 \in \mathbb{H}$. Show that the perpendicular bisector of $[z_1, z_2]$ can also be written as $\{z \in \mathbb{H} \mid y_2|z - z_1|^2 = y_1|z - z_2|^2\}$.

Solution:
By the previous Proposition, $z \in \mathbb{H}$ is on the perpendicular bisector of $[z_1, z_2]$ if and only if $d_{\mathbb{H}}(z, z_1) = d_{\mathbb{H}}(z, z_2)$.
Note that:

$$\begin{aligned} d_{\mathbb{H}}(z, z_1) &= d_{\mathbb{H}}(z, z_2) \\ \cosh d_{\mathbb{H}}(z, z_1) &= \cosh d_{\mathbb{H}}(z, z_2) \\ 1 + \frac{|z - z_1|^2}{2y_1 \mathrm{Im}(z)} &= 1 + \frac{|z - z_2|^2}{2y_2 \mathrm{Im}(z)} \\ y_2|z - z_1|^2 &= y_1|z - z_2|^2 \end{aligned}$$

Tools for Dirichlet polygon

Theorem

Let Γ be a non-trivial Fuchsian group. Then there exists a point $p \in \mathbb{H}$ that is not a fixed point for any non-trivial element of Γ. (That is, $\gamma(p) \neq p$ for all $\gamma \in \Gamma \setminus \{\mathrm{Id}\}$.)

Tools for Dirichlet polygon - continued

Definition: Let Γ be a Fuchsian group and let $p \in \mathbb{H}$ be a point such that $\gamma(p) \neq p$ for all $\gamma \in \Gamma \setminus \{\mathrm{Id}\}$. Let γ be an element of Γ and suppose that γ is not the identity. The set

$$\{z \in \mathbb{H} \,|\, d_{\mathbb{H}}(z,p) < d_{\mathbb{H}}(z,\gamma(p))\}$$

consists of all points $z \in \mathbb{H}$ that are closer to p than to $\gamma(p)$.

Definition: We define the Dirichlet region to be:

$$D(p) = \{z \in \mathbb{H} \,|\, d_{\mathbb{H}}(z,p) < d_{\mathbb{H}}(z,\gamma(p)) \text{ for all } \gamma \in \Gamma \setminus \{\mathrm{Id}\}\}$$

Thus the Dirichlet region is the set of all points z that are closer to p than to any other point in the orbit $\Gamma(p) = \{\gamma(p) \,|\, \gamma \in \Gamma\}$ of p under Γ.

Tools for Dirichlet polygon - continued

Fact: Let Γ be a Fuchsian group and let p be a point not fixed by any non-trivial element of Γ. Then the Dirichlet region $D(p)$ is a fundamental domain for Γ. Moreover, if $\text{Area}_{\mathbb{H}}(D(p)) < \infty$ then $D(p)$ is a convex hyperbolic polygon; in particular it has finitely many edges.

Remark 1: There are many other hypotheses that ensure that $D(p)$ is a convex hyperbolic polygon with finitely many edges; requiring $D(p)$ to have finite hyperbolic area is probably the simplest. Fuchsian groups that have a convex hyperbolic polygon with finitely many edges as a Dirichlet region are called **geometrically** finite.

Tools for Dirichlet polygon - continued

Remark 2: If $D(p)$ has finitely many edges then we refer to $D(p)$ as a Dirichlet polygon. Notice that some of these edges may be arcs of $\partial\mathbb{H}$. If there are finitely many edges then there are also finitely many vertices (some of which may be on $\partial\mathbb{H}$).

Remark 3: The Dirichlet polygon $D(p)$ depends on p. If we choose a different point p, then we may obtain a different polygon with different properties, such as the number of edges.

Summary: procedure to construct a Dirichlet polygon for a given Fuchsian group

1. Choose $p \in \mathbb{H}$ such that $\gamma(p) \neq p$, $\forall \gamma \in \Gamma \setminus \{id\}$.
2. Let $\gamma \in \Gamma \setminus \{id\}$. Construct the geodesic segment $[p, \gamma(p)]$.
3. Let $L_p(\gamma)$ denote the perpendicular bisector of $[p, \gamma(p)]$.
4. Let $H_p(\gamma)$ denote the half-plane determined by $L_p(\gamma)$ that contains p.
5. Let

$$D(p) = \bigcap_{\gamma \in \Gamma \setminus \{id\}} H_p(\gamma)$$

Example (I): The group of all integer translations

Let Γ be the Fuchsian group $\{\gamma_n \,|\, \gamma_n(z) = z + n, n \in \mathbb{Z}\}$. Then $D(i) = \{z \in \mathbb{H} \,|\, -\frac{1}{2} < Re(z) < \frac{1}{2}\}$.

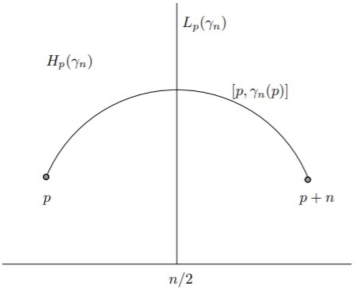
Solution:
Let $p = i$. Then clearly $\gamma_n(p) = i + n \neq p$ so that p is not fixed by any non-trivial element of Γ. As $\gamma_n(p) = i + n$, it is clear that the perpendicular bisector of $[p, \gamma_n(p)]$ is the vertical straight line with real part $\frac{n}{2}$. Hence,

$$H_p(\gamma_n) = \begin{cases} \{z \in \mathbb{H} \,|\, Re(z) < n/2\} & \text{if } n > 0 \\ \{z \in \mathbb{H} \,|\, Re(z) > n/2\} & \text{if } n < 0 \end{cases}$$

Example (I) - continued

Hence,

$$
\begin{aligned}
D(p) &= \bigcap_{\gamma \in \Gamma \setminus \{\mathrm{Id}\}} H_p(\gamma) \\
&= H_p(\gamma_1) \cap H_p(\gamma_{-1}) \\
&= \{z \in \mathbb{H} \mid -1/2 < Re(z) < 1/2\}
\end{aligned}
$$



Example (II)

Let $= \{\gamma_n \,|\, \gamma_n(z) = 2^n z, n \in \mathbb{Z}\}$. This is a Fuchsian group. Choose a suitable $p \in \mathbb{H}$ and construct a Dirichlet polygon $D(p)$.

Solution: Let $\Gamma = \{\gamma_n \,|\, \gamma_n(z) = 2^n z\}$. Let $p = i$ and note that $\gamma_n(p) = 2^n i \neq p$ unless $n = 0$. For each n, $[p, \gamma_n(p)]$ is the arc of imaginary axis from i to $2^n i$. Suppose first that $n > 0$. Recalling that for $a < b$ we have $d_{\mathbb{H}}(ai, bi) = log\, b/a$ it is easy to see that the midpoint of $[i, 2^n i]$ is at $2^{n/2} i$.

Proof - continued

Hence, $L_p(\gamma_n)$ is the semicircle of radius $2^{n/2}$ centred at the origin and

$$H_p(\gamma_n) = \{z \in \mathbb{H} \,|\, |z| < 2^{n/2}\}$$

For $n < 0$, we can see that

$$H_p(\gamma_n) = \{z \in \mathbb{H} \,|\, |z| > 2^{n/2}\}$$

Hence,

$$\begin{aligned} D(p) &= \bigcap_{\gamma_n \in \Gamma \setminus \{\mathrm{Id}\}} H_p(\gamma_n) \\ &= \{z \in \mathbb{H} \,|\, 1/\sqrt{2} < |z| < \sqrt{2}\} \end{aligned}$$

Outline

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